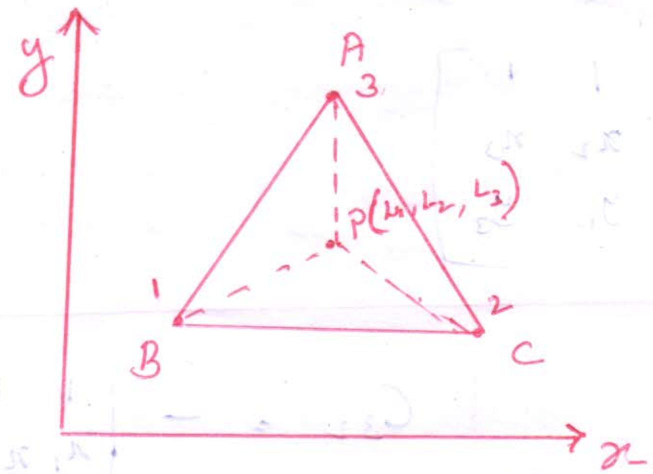
MODULE. 3

Natural Co-ordinates in 2D..



Let's Consider a triangular element having 3 nodes as shown in figure. P be a point inside the element at it has 3 coordinates $L_1, L_2 \& L_3$

From the definition of natural coordinates, we know that

$$\left.\begin{aligned} L_1 + L_2 + L_3 &= 1 \\ L_1 x_1 + L_2 x_2 + L_3 x_3 &= x \\ L_1 y_1 + L_2 y_2 + L_3 y_3 &= y \end{aligned}\right\} \rightarrow (1)$$

representing in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$D^{-1} = \frac{C^T}{|D|}$$

$$C_{11} = + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} = x_2 y_3 - x_3 y_2$$

$$C_{12} = - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} = x_3 y_1 - x_1 y_3$$

$$C_{13} = + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

$$C_{21} = - \begin{vmatrix} 1 & 1 \\ y_2 & y_1 \end{vmatrix} = y_2 - y_1$$

$$C_{22} = + \begin{vmatrix} 1 & 1 \\ y_1 & y_3 \end{vmatrix} = y_3 - y_1$$

$$C_{23} = - \begin{vmatrix} 1 & 1 \\ y_1 & y_2 \end{vmatrix} = y_1 - y_2$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ x_2 & x_3 \end{vmatrix} = x_3 - x_2$$

$$C_{32} = - \begin{vmatrix} 1 & 1 \\ x_1 & x_3 \end{vmatrix} = x_1 - x_3$$

$$C_{33} = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1.$$

$$|D| = (x_2 y_3 - x_3 y_2) - 1 (x_1 y_3 - x_3 y_1) + 1 (x_1 y_2 - x_2 y_1)$$

$$C = \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}$$

We need C^T.

$$D^{-1} = \frac{C^T}{|D|}$$

$$= \frac{1}{(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \quad " \quad " \quad \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \rightarrow (2)$$

the area of triangle ABC can be given by.

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$A = \frac{1}{2} \left[x_2y_3 - x_3y_2 - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1) \right]$$

$$2A = (x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1) \rightarrow (3)$$

Sub: 3 in (2) we get.

$$\boxed{\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}}$$

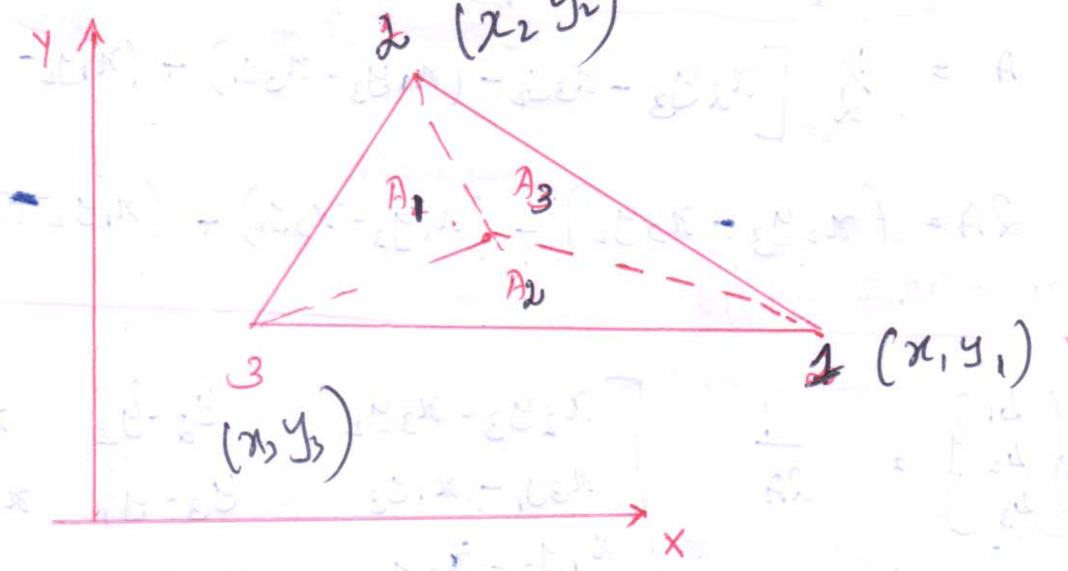
Integration of polynomials term in natural co-ords. for 2D elements can

$$\boxed{\oint_A (L_1)^\alpha (L_2)^\beta (L_3)^\gamma \, dA = \frac{\alpha!\,\beta!\,\gamma!}{(\alpha+\beta+\gamma+2)!} \times 2A}$$

Shape functions using Area Co-ordinates.

The interpolation functions for a triangular element are algebraically complex if expressed in Cartesian Co-ordinates, more over, the integration required to obtain the element stiffness matrix is complex.

The interpolation function & subsequently the required integration can be obtained in a simplified manner by the Concept of area Coordinates.



Considering a linear displacement variation of a triangular element is shown in above fig. the displacement at any point can be written as.

$$u = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3.$$

(or)

$$u = \{\phi\}^T \{\alpha\} \quad \rightarrow (1)$$

$$L_1 = \frac{A_1}{A}, \quad L_2 = \frac{A_2}{A}, \quad L_3 = \frac{A_3}{A}.$$

$A \rightarrow$ area of triangle 1, 2, 3.

$$A_1 + A_2 + A_3 = A.$$

$$L_1 + L_2 + L_3 = 1.$$

$$L_1 = \frac{A_1}{A} = \frac{2A_1}{2A} = \frac{\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}$$

$$L_1 = \frac{1}{2A}\left[(x_2 y_3 - y_2 x_3) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$

$$2A = \left[(x_2 y_3 - y_2 x_3) + (x_3 y_1 - y_3 x_1) + (x_1 y_2 - y_1 x_2)\right]$$

L_2 & L_3 can be obtained similarly.

$$L_2 = \frac{1}{2A} \begin{vmatrix} 1 & x & y \\ 1 & x_3 & y_3 \\ 1 & x_1 & y_1 \end{vmatrix}$$

$$= \frac{1}{2A}\left[(x_3 y_1 - y_3 x_1) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$

$$L_3 = \frac{1}{2A} \begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}$$

$$= \frac{1}{2A}\left[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]$$

In gener the area Co-ordinates can be written as

$$L_i = a_i + b_i x + c_i y$$

$$a_i = (x_j y_k - y_j x_k)/2A.$$

$$b_i = (y_j - y_k)/2A$$

$$c_i = (x_k - x_j)/2A.$$

for $i = 1, j = 2, k = 3$

$i = 2, j = 3, k = 1$

$i = 3, j = 1$ & $k = 2$.

at node 1, $L_1 = 1$, $L_2 = L_3 = 0$.

$L_2 = 1$, $L_1 = L_3 = 0$

$L_3 = 1$, $L_1 = L_2 = 0$.

$$\{\phi\}^T = [L_1 \; L_2 \; L_3] + \{\alpha_1 \; \alpha_2 \; \alpha_3\} = \{\alpha\}^T$$

$$L_1 = \frac{A_1}{A}, \quad L_2 = \frac{A_2}{A}, \quad L_3 = \frac{A_3}{A}.$$

we know that $L_1 + L_2 + L_3 = 1$.

at node 1, $L_1 = 1$, $L_2 = L_3 = 0$.

" 2 $L_2 = 1$, $L_1 = L_3 = 0$.

" 3 $L_3 = 1$, $L_1 = L_2 = 0$.

$$u = \{\phi^T\} \cdot \{\alpha\}$$

$$u_i = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_1 & L_2 & L_3 \\ L_1 & L_2 & L_3 \end{bmatrix} \cdot \{\alpha\}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{\alpha\}$$

$$\{\alpha\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \longrightarrow (2)$$

Sub: (2) in (1).

$$\cancel{u = [L_1 \; L_2 \; L_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}}$$

$$\{u\} = \{\phi\}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \{q\}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{u_i\}$$

The above expression can be written as

$$u = \{N\}^T \{u_i\}$$

$$\{N\}^T = [L_1 \; L_2 \; L_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [L_1 \; L_2 \; L_3]$$

$$u = L_1 u_1 + L_2 u_2 + L_3 u_3$$

III $V = L_1 v_1 + L_2 v_2 + L_3 v_3$

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 & L_2 & L_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$[N] = \begin{bmatrix} L_1 & L_2 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 & L_2 & L_3 \end{bmatrix}$$

(or)

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} L_1 & 0 & L_2 & 0 & L_3 & 0 \\ 0 & L_1 & 0 & L_2 & 0 & L_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

so then

$u = N_1 u_1 + N_2 u_2 + N_3 u_3$

$V = N_1 v_1 + N_2 v_2 + N_3 v_3$